Chapter 6: Systems of Particles and Rotational Motion

NCERT Exercise Questions & Answers

Question 6.1

Give the location of the centre of mass of a (i) sphere, (ii) cylinder, (iii) ring, and (iv) cube, each of uniform mass density. Does the centre of mass of a body necessarily lie inside the body?

Answer:

For bodies with uniform mass density and regular geometric shapes:

(i) Sphere: Centre of mass is at the geometric center (center of the sphere)

(ii) Cylinder: Centre of mass is at the geometric center (midpoint of the axis of symmetry)

(iii) Ring: Centre of mass is at the geometric center (center of the ring)

(iv) Cube: Centre of mass is at the geometric center (intersection of all diagonals)

Does CM necessarily lie inside the body? No, the centre of mass does not necessarily lie inside the body.

Examples:

- For a ring, the CM is at the center which is empty space
- For a hollow sphere, the CM is at the center (hollow region)

- For an L-shaped object, the CM may lie outside the material of the body

The CM represents the average position of mass distribution and can be located in regions without any actual mass.

Question 6.2

In the HCl molecule, the separation between the nuclei of the two atoms is about 1.27 Å (1 Å = 10^{-10} m). Find the approximate location of the CM of the molecule, given that a chlorine atom is about 35.5 times as massive as a hydrogen atom and nearly all the mass of an atom is concentrated in its nucleus.

Answer:

Given data:

- Separation between nuclei = 1.27 Å = 1.27×10^{-10} m
- Mass of Cl atom = 35.5 × mass of H atom
- Let mass of H = m, then mass of Cl = 35.5m

Taking H atom at origin (x = 0) and Cl atom at x = 1.27×10^{-10} m:

Using the center of mass formula:

$$X\_CM = (m_1x_1 + m_2x_2)/(m_1 + m_2)$$

$$X\_CM = (m \times 0 + 35.5m \times 1.27 \times 10^{-10})/(m + 35.5m)$$

$$X\_CM = (35.5 \times 1.27 \times 10^{-10})/36.5 \text{ m}$$

$$X\_CM = 45.085 \times 10^{-10}/36.5 = 1.235 \times 10^{-10} \text{ m}$$

X_CM ≈ 1.23 Å from the hydrogen atom

The CM is located very close to the chlorine atom due to its much larger mass.

Question 6.3

A child sits stationary at one end of a long trolley moving uniformly with a speed V on a smooth horizontal floor. If the child gets up and runs about on the trolley in any manner, what is the speed of the CM of the (trolley + child) system?

Answer:

The speed of the CM remains V (unchanged).

Explanation:

- Initially, both trolley and child move with speed V
- The CM of the system moves with speed V
- When the child runs on the trolley, these are internal forces within the system
- No external horizontal forces act on the system (smooth floor)
- According to the principle of conservation of momentum and motion of CM:

$$M\_total \times V\_CM = \text{constant}$$

Since there are no external forces, the CM continues to move with the same speed V, regardless of the child's motion relative to the trolley.

The child's running only redistributes the internal momentum between child and trolley, but doesn't change the total momentum or CM motion.

Question 6.4

Show that the area of the triangle contained between the vectors a and b is one half of the magnitude of a × b.

Answer:

Consider two vectors **a** and **b** originating from a common point O, making an angle θ between them.

The triangle formed has:

- Base = |**a**|
- Height = |**b**| sin θ (perpendicular distance from tip of **b** to line of **a**)

Area of triangle = ½ × base × height

$$\text{Area} = \frac{1}{2} \times |a| \times |b| \sin\theta$$

From the definition of cross product:

$$|a \times b| = |a|\,|b| \sin\theta$$

Therefore:

$$\text{Area of triangle} = \frac{1}{2}|a \times b|$$

Hence proved that the area of triangle formed by vectors **a** and **b** is half the magnitude of their cross product.

Question 6.5

Show that a·(b × c) is equal in magnitude to the volume of the parallelepiped formed on the three vectors a, b and c.

Answer:

Consider three vectors **a**, **b**, and **c** forming a parallelepiped.

The volume of a parallelepiped is given by:

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Volume = Base area × Height
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Step 1: Base area The base is formed by vectors **b** and **c**. Base area = |**b** × **c**| (magnitude of cross product gives area of parallelogram)

Step 2: Height The height is the component of vector **a** perpendicular to the base plane containing **b** and **c**.

If **n̂** is the unit normal to the base plane:

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n̂ = (b × c)/|b × c|
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Height = component of **a** along **n̂** = **a**·**n̂** = **a**·((**b** × **c**)/|**b** × **c**|)

Step 3: Volume calculation

$$\text{Volume} = |b \times c| \times |a\cdot(b \times c)|/|b \times c|$$
$$\text{Volume} = |a\cdot(b \times c)|$$

Therefore, **|a·(b × c)| = Volume of parallelepiped**

Note: a·(b × c) is called the scalar triple product.

Question 6.6

Find the components along the x, y, z axes of the angular momentum l of a particle, whose position vector is r with components x, y, z and momentum is p with components p_x, p_y, p_y. Show that if the particle moves only in the x-y plane the angular momentum has only a z-component.

Answer:

Given: **r** = $x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ and **p** = $p_x\hat{\mathbf{i}} + p_y\hat{\mathbf{j}} + p_y\hat{\mathbf{k}}$

Angular momentum **l** = **r** × **p**

Using determinant form:

$$l = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_y \end{vmatrix}$$

Components:

$l_x = yp_y - zp_y$
$l_y = zp_x - xp_y$
$l_y = xp_y - yp_x$

For motion in x-y plane only:

- z = 0 (particle confined to x-y plane)
- p_y = 0 (no momentum component along z-axis)

Substituting:

$l_x = y(0) - 0(p_y) = 0$
$l_y = 0(p_x) - x(0) = 0$
$l_y = xp_y - yp_x \neq 0$

Therefore, for motion in x-y plane, angular momentum has only z-component:

$l = (xp_y - yp_x)\hat{k}$

Question 6.7

Two particles, each of mass m and speed v, travel in opposite directions along parallel lines separated by a distance d. Show that the angular momentum vector of the two particle system is the same whatever be the point about which the angular momentum is taken.

Answer:

Let the particles move along parallel lines separated by distance d.

Case 1: Angular momentum about point O_1 Let particle 1 be at distance r_1 from O_1 and particle 2 at distance r_2 from O_1.

For particle 1: $\mathbf{l_1} = \mathbf{r_1} \times \mathbf{p_1} = \mathbf{r_1} \times m\mathbf{v_1}$ For particle 2: $\mathbf{l_2} = \mathbf{r_2} \times \mathbf{p_2} = \mathbf{r_2} \times m\mathbf{v_2}$

Since velocities are equal in magnitude but opposite in direction, and the geometry is such that the perpendicular distances contribute equally:

Total angular momentum: $\mathbf{L_1} = \mathbf{l_1} + \mathbf{l_2}$

Case 2: Angular momentum about point O_2 (different from O_1) Let $O_1O_2 = \mathbf{a}$

New position vectors: $\mathbf{r_1'} = \mathbf{r_1} - \mathbf{a}$ and $\mathbf{r_2'} = \mathbf{r_2} - \mathbf{a}$

$\mathbf{L_2} = \mathbf{r_1'} \times m\mathbf{v_1} + \mathbf{r_2'} \times m\mathbf{v_2}$ $\mathbf{L_2} = (\mathbf{r_1} - \mathbf{a}) \times m\mathbf{v_1} + (\mathbf{r_2} - \mathbf{a}) \times m\mathbf{v_2}$ $\mathbf{L_2} = \mathbf{r_1} \times m\mathbf{v_1} + \mathbf{r_2} \times m\mathbf{v_2} - \mathbf{a} \times m(\mathbf{v_1} + \mathbf{v_2})$

Since $\mathbf{v_1} = -\mathbf{v_2}$ (opposite directions), $\mathbf{v_1} + \mathbf{v_2} = \mathbf{0}$

Therefore: $\mathbf{L_2} = \mathbf{r_1} \times m\mathbf{v_1} + \mathbf{r_2} \times m\mathbf{v_2} = \mathbf{L_1}$

Hence, the angular momentum is independent of the choice of origin.

Question 6.8

A non-uniform bar of weight W is suspended at rest by two strings of negligible weight as shown in Fig.6.33. The angles made by the strings with the vertical are 36.9° and 53.1° respectively. The bar is 2 m long. Calculate the distance d of the centre of gravity of the bar from its left end.

Answer:

Given:

- Weight of bar = W
- Length of bar = 2 m
- String angles with vertical: 36.9° and 53.1°
- String tensions: T_1 and T_2

Step 1: Force equilibrium (vertical)

$$T_1 \cos(36.9°) + T_2 \cos(53.1°) = W$$
$$T_1(0.8) + T_2(0.6) = W \quad \dots (1)$$

Step 2: Force equilibrium (horizontal)

$$T_1 \sin(36.9°) = T_2 \sin(53.1°)$$
$$T_1(0.6) = T_2(0.8)$$
$$T_1 = (4/3)T_2 \quad \dots (2)$$

Step 3: Substitute (2) into (1)

$$(4/3)T_2(0.8) + T_2(0.6) = W$$
$$(32/30)T_2 + (18/30)T_2 = W$$
$$(50/30)T_2 = W$$
$$T_2 = 0.6W \text{ and } T_1 = 0.8W$$

Step 4: Torque equilibrium about left end Let d = distance of CG from left end Distance of right string from left end = 2 m

Taking torques about left end:

$W \times d = T_2 \times 2 \times \sin(53.1°)$
$W \times d = 0.6W \times 2 \times 0.8$
$d = 0.96$ m

Distance of centre of gravity from left end = 0.96 m

Question 6.9

A car weighs 1800 kg. The distance between its front and back axles is 1.8 m. Its centre of gravity is 1.05 m behind the front axle. Determine the force exerted by the level ground on each front wheel and each back wheel.

Answer:

Given:

- Weight of car = 1800 kg
- Distance between axles = 1.8 m
- CG is 1.05 m behind front axle
- Distance of CG from rear axle = 1.8 - 1.05 = 0.75 m

Let:

- Force on each front wheel = F_f
- Force on each back wheel = F_b
- Total force on front axle = $2F_f$
- Total force on rear axle = $2F_b$

Force equilibrium (vertical):

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2F_f + 2F_b = 1800g
F_f + F_b = 900g  ... (1)
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Torque equilibrium about front axle:

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1800g × 1.05 = 2F_b × 1.8
F_b = (1800 × 1.05)/(2 × 1.8) = 525 N
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From equation (1):

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F_f = 900g - F_b = 900g - 525 = 375g N
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Taking g = 9.8 m/s^2:

- **Force on each front wheel = 375 × 9.8 = 3675 N**
- **Force on each back wheel = 525 × 9.8 = 5145 N**

Question 6.10

Torques of equal magnitude are applied to a hollow cylinder and a solid sphere, both having the same mass and radius. The cylinder is free to rotate about its standard axis of symmetry, and the sphere is free to rotate about an axis passing through its centre. Which of the two will acquire a greater angular speed after a given time?

Answer:

Given: Equal torques applied for equal time to both objects

From $\tau = I\alpha$, we have $\alpha = \tau/I$

For constant torque: $\omega = \alpha t = (\tau/I)t$

Moments of inertia:

- Hollow cylinder about its axis: $I_{cylinder} = MR^2$
- Solid sphere about diameter: $I_{sphere} = (2/5)MR^2$

Angular accelerations:

- $\alpha_{cylinder} = \tau/(MR^2)$
- $\alpha_{sphere} = \tau/((2/5)MR^2) = 5\tau/(2MR^2)$

Angular velocities after time t:

- $\omega_{cylinder} = (\tau/MR^2)t$
- $\omega_{sphere} = (5\tau/2MR^2)t$

Comparing:

$$\omega_{sphere}/\omega_{cylinder} = (5\tau/2MR^2)t \,/\, (\tau/MR^2)t = 5/2 = 2.5$$

The solid sphere will acquire 2.5 times greater angular speed than the hollow cylinder.

Reason: The sphere has smaller moment of inertia, so the same torque produces greater angular acceleration.

Question 6.11

A solid cylinder of mass 20 kg rotates about its axis with angular speed 100 rad s^{-1}. The radius of the cylinder is 0.25 m. What is the kinetic energy associated with the rotation of the cylinder? What is the magnitude of angular momentum of the cylinder about its axis?

Answer:

Given:

- Mass $M = 20$ kg
- Angular speed $\omega = 100$ rad/s
- Radius $R = 0.25$ m

Moment of inertia of solid cylinder:

$$I = (1/2)MR^2 = (1/2) \times 20 \times (0.25)^2 = 0.625\ \text{kg·m}^2$$

Rotational kinetic energy:

$$K = (1/2)I\omega^2$$
$$K = (1/2) \times 0.625 \times (100)^2$$
$$K = 0.3125 \times 10000 = 3125\ \text{J}$$

Angular momentum:

$$L = I\omega$$
$$L = 0.625 \times 100 = 62.5\ \text{kg·m}^2\text{·s}^{-1}$$

Answers:

- **Kinetic energy = 3125 J**
- **Angular momentum = 62.5 kg·m²·s⁻¹**

Question 6.12

(a) A child stands at the centre of a turntable with his two arms outstretched. The turntable is set rotating with an angular speed of 40 rev/min. How much is the angular speed of the child if he folds his hands back and thereby reduces his moment of inertia to 2/5 times the initial value? Assume that the turntable rotates without friction.

(b) Show that the child's new kinetic energy of rotation is more than the initial kinetic energy of rotation. How do you account for this increase in kinetic energy?

Answer:

(a) Finding new angular speed:

Given:

- Initial angular speed $\omega_1 = 40$ rev/min
- Final moment of inertia $I_2 = (2/5)I_1$
- No friction (no external torque)

Applying conservation of angular momentum:

$$L_1 = L_2$$
$$I_1\omega_1 = I_2\omega_2$$
$$I_1 \times 40 = (2/5)I_1 \times \omega_2$$
$$\omega_2 = 40 \times 5/2 = 100 \text{ rev/min}$$

New angular speed = 100 rev/min

(b) Comparing kinetic energies:

Initial kinetic energy:

$$K_1 = (1/2)I_1\omega_1^2$$

Final kinetic energy:

$$K_2 = (1/2)I_2\omega_2^2 = (1/2) \times (2/5)I_1 \times (100)^2$$
$$K_2 = (1/5)I_1 \times 10000 = 2000I_1$$

Comparing with K_1:

$$K_1 = (1/2)I_1 \times (40)^2 = 800I_1$$

Ratio:

$$K_2/K_1 = 2000I_1/800I_1 = 2.5$$

$K_2 = 2.5K_1$, so the final kinetic energy is 2.5 times the initial value.

Explanation for increase: The child does work against centrifugal force when pulling his arms inward. This muscular work gets converted into additional rotational kinetic energy. Energy is not violated; the child's internal energy decreases by the amount of work done.

Question 6.13

A rope of negligible mass is wound round a hollow cylinder of mass 3 kg and radius 40 cm. What is the angular acceleration of the cylinder if the rope is pulled with a force of 30 N? What is the linear acceleration of the rope? Assume that there is no slipping.

Answer:

Given:

- Mass of hollow cylinder $M = 3$ kg
- Radius $R = 40$ cm $= 0.4$ m
- Applied force $F = 30$ N
- No slipping condition

Moment of inertia of hollow cylinder:

$$I = MR^2 = 3 \times (0.4)^2 = 0.48 \text{ kg·m}^2$$

Torque applied:

$$\tau = F \times R = 30 \times 0.4 = 12 \text{ N·m}$$

Angular acceleration:

$$\alpha = \tau/I = 12/0.48 = 25 \text{ rad/s}^2$$

Linear acceleration of rope: For no slipping condition: $a = \alpha R$

$$a = 25 \times 0.4 = 10 \text{ m/s}^2$$

Answers:

- **Angular acceleration = 25 rad/s^2**
- **Linear acceleration of rope = 10 m/s^2**

Question 6.14

To maintain a rotor at a uniform angular speed of 200 rad s^{-1}, an engine needs to transmit a torque of 180 N m. What is the power required by the engine? (Note: uniform angular velocity in the absence of friction implies zero torque. In practice, applied torque is needed to counter frictional torque). Assume that the engine is 100% efficient.

Answer:

Given:

- Angular speed ω = 200 rad/s (constant)
- Torque τ = 180 N·m (to overcome friction)
- Engine efficiency = 100%

Power in rotational motion:

$$P = \tau\omega$$
$$P = 180 \times 200 = 36{,}000 \text{ W} = 36 \text{ kW}$$

Power required by the engine = 36 kW

Note: The torque is needed to balance the frictional torque to maintain constant angular velocity. At steady state, net torque is zero, but the engine must continuously supply power to overcome energy losses due to friction.

Question 6.15

From a uniform disk of radius R, a circular hole of radius R/2 is cut out. The centre of the hole

is at R/2 from the centre of the original disc. Locate the centre of gravity of the resulting flat body.

Answer:

Method: Treat as combination of positive and negative masses

Let the original disk have mass M.

Original complete disk:

- Mass = M
- CM at center (0, 0)

Removed portion (hole):

- Radius = R/2
- Area = $\pi(R/2)^2 = \pi R^2/4$
- Original disk area = πR^2
- Mass of hole = $M \times (\pi R^2/4)/(\pi R^2) = M/4$
- CM of hole at distance R/2 from center

Remaining body:

- Mass = M - M/4 = 3M/4

Applying CM formula: Taking center of original disk as origin, with hole center at (R/2, 0):

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X_CM = (M × 0 - (M/4) × (R/2))/(3M/4)
X_CM = (-MR/8)/(3M/4) = -R/6
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The center of gravity is at distance R/6 from the original center, on the side opposite to the hole.

Question 6.16

A metre stick is balanced on a knife edge at its centre. When two coins, each of mass 5 g are put one on top of the other at the 12.0 cm mark, the stick is found to be balanced at 45.0 cm. What is the mass of the metre stick?

Answer:

Given:

- Length of stick = 100 cm
- Mass of each coin = 5 g
- Total mass of coins = 10 g
- Coins placed at 12.0 cm mark
- New balance point at 45.0 cm mark

Let mass of stick = M

Initially: Stick balanced at 50 cm (center)

Finally: System balanced at 45.0 cm

Taking torques about the 45.0 cm mark:

Torque due to stick: CM of stick is at 50 cm mark Distance from balance point = 50 - 45 = 5 cm Torque = M × g × 5 (clockwise)

Torque due to coins: Coins at 12.0 cm mark

Distance from balance point = 45 - 12 = 33 cm Torque = 10 × g × 33 (anticlockwise)

For equilibrium:

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M × 5 = 10 × 33
M = 330/5 = 66 g
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Mass of the metre stick = 66 g

Question 6.17

The oxygen molecule has a mass of 5.30×10^{-26} kg and a moment of inertia of 1.94×10^{-46} kg m^2 about an axis through its centre perpendicular to the lines joining the two atoms. Suppose the mean speed of such a molecule in a gas is 500 m/s and that its kinetic energy of rotation is two thirds of its kinetic energy of translation. Find the average angular velocity of the molecule.

Answer:

Given:

- Mass m = 5.30×10^{-26} kg
- Moment of inertia I = 1.94×10^{-46} kg·m^2
- Mean speed v = 500 m/s
- K_rotational = (2/3) × K_translational

Translational kinetic energy:

$$K\_trans = (1/2)mv^2 = (1/2) \times 5.30 \times 10^{-26} \times (500)^2$$
$$K\_trans = 6.625 \times 10^{-21}\ J$$

Rotational kinetic energy:

$$K\_rot = (2/3) \times K\_trans = (2/3) \times 6.625 \times 10^{-21}$$
$$K\_rot = 4.417 \times 10^{-21}\ J$$

Angular velocity:

$$K\_rot = (1/2)I\omega^2$$
$$4.417 \times 10^{-21} = (1/2) \times 1.94 \times 10^{-46} \times \omega^2$$
$$\omega^2 = (2 \times 4.417 \times 10^{-21})/(1.94 \times 10^{-46})$$
$$\omega^2 = 4.55 \times 10^{25}$$
$$\omega = 2.13 \times 10^{12}\ rad/s$$

Average angular velocity = 2.13×10^{12} rad/s