Chapter 7: Gravitation - Detailed Notes

7.1 INTRODUCTION

Historical Background:

- **Galileo (1564-1642)**: First to recognize that all bodies fall toward Earth with constant acceleration, regardless of mass
- Conducted experiments with inclined planes to measure gravitational acceleration
- Value obtained was close to modern accurate measurements

Early Models of Planetary Motion:

1. Geocentric Model (Ptolemy - 2000 years ago)

- Earth at center of universe
- All celestial objects (stars, sun, planets) revolve around Earth
- Only circular motion considered possible for celestial objects
- Complex system of circles within circles to explain planetary motion

2. Heliocentric Model Development

- **Aryabhatta (5th century AD)**: Early mention of sun-centered model in Indian astronomy
- **Nicolas Copernicus (1473-1543)**: Proposed definitive heliocentric model
- Planets move in circles around fixed central sun
- Model was opposed by church; Galileo faced prosecution for supporting it

3. Observational Revolution

- **Tycho Brahe (1546-1601)**: Compiled precise naked-eye observations of planetary positions
- **Johannes Kepler (1571-1640)**: Analyzed Brahe's data to derive three laws of planetary motion
- Kepler's laws provided foundation for Newton's universal gravitation theory

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7.2 KEPLER'S LAWS

Law 1: Law of Orbits

All planets move in elliptical orbits with the Sun at one of the foci

Ellipse Properties:

- **Foci**: Two fixed points F_1 and F_2
- **Definition**: For any point on ellipse, sum of distances from both foci is constant
- **Semi-major axis**: Half the distance between farthest points (PO = AO)
- **Special case**: Circle (when both foci merge into one point)

Construction Method:

1. Fix string ends at two points (foci)
2. Keep string taut with pencil
3. Draw curve by moving pencil
4. Result: elliptical orbit

Planetary Terms:

- **Perihelion (P)**: Closest point to Sun
- **Aphelion (A)**: Farthest point from Sun

Law 2: Law of Areas

The radius vector from Sun to planet sweeps equal areas in equal time intervals

Mathematical Expression:

$$\Delta A/\Delta t = \text{constant}$$

Physical Interpretation:

- Planets move faster when closer to Sun (at perihelion)
- Planets move slower when farther from Sun (at aphelion)
- Consequence of angular momentum conservation for central forces

Derivation from Angular Momentum:

$$\Delta A = \tfrac{1}{2}(r\square \times v\square\Delta t)$$
$$\Delta A/\Delta t = \tfrac{1}{2}(r\square \times p\square)/m = L/(2m)$$

Since L is constant for central force, $\Delta A/\Delta t$ is constant

Law 3: Law of Periods

The square of orbital period is proportional to cube of semi-major axis

Mathematical Form:

$$T^2 \propto a^3$$

Where T = period, a = semi-major axis

Verification Data:

Planet	Semi-major axis (10^{10} m)	Period (years)	T^2/a^3 (10^{-34} y^2m^{-3})
Mercury	5.79	0.24	2.95
Venus	10.8	0.615	3.00
Earth	15.0	1	2.96
Mars	22.8	1.88	2.98

Key Insight: The ratio T^2/a^3 is approximately constant for all planets

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7.3 UNIVERSAL LAW OF GRAVITATION

Newton's Reasoning:

- Moon's orbital motion requires centripetal acceleration
- Centripetal acceleration: $a_m = v^2/R_m = 4\pi^2 R_m/T^2$
- With T ≈ 27.3 days, $R_m \approx 3.84 \times 10^8$ m
- Calculated a_m much smaller than surface gravity g
- Concluded gravitational force decreases with distance

Distance Dependence Analysis:

If gravitational force $\propto 1/r^2$, then:

$$a_m/g = (R_E/R_m)^2 \approx 1/3600$$

This matches observed ratio, confirming inverse square law

Newton's Universal Law of Gravitation:

Statement:

Every body in the universe attracts every other body with force directly proportional to product of their masses and inversely proportional to square of distance between them

Mathematical Form:

$$F = Gm_1m_2/r^2$$

Vector Form:

$$\vec{F}_{12} = -Gm_1m_2\hat{r}_{12}/r^2$$
$$\vec{F}_{12} = -Gm_1m_2\vec{r}_{12}/r^3$$

Where:

- G = Universal gravitational constant
- $\hat{r}_{12}$ = unit vector from m_1 to m_2
- Force is attractive (along $-\hat{r}$)

Superposition Principle:

For multiple masses, total force on m_1:

$$\vec{F}\_total = \Sigma\vec{F}_i = \Sigma(Gm_1m_i\hat{r}_i/r_i^2)$$

Extended Objects:

For extended objects (not point masses):

1. **Hollow spherical shell + external point**: Force acts as if all shell mass concentrated at center

2. **Hollow spherical shell + internal point**: Net gravitational force is zero

Physical reasoning: Various shell regions exert forces in different directions that cancel (internal) or combine along radial direction (external)

7.4 THE GRAVITATIONAL CONSTANT

Cavendish Experiment (1798):

Apparatus:

- Light rod with small lead spheres at ends
- Rod suspended by fine wire (torsion balance)
- Large lead spheres brought near small ones on opposite sides
- System measures gravitational attraction between spheres

Working Principle:

1. Gravitational force between spheres creates torque
2. Wire twists until restoring torque balances gravitational torque
3. Angle of twist θ measured
4. Restoring torque: $\tau = c\theta$ (c = torsional constant)

Force Calculation:

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F = GMm/d²
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Where M, m = masses of spheres, d = center-to-center distance

Equilibrium Condition:

$$GLMm/d^2 = c\theta/L$$

Solving for G:

$$G = c\theta d^2/(MLm)$$

Modern Value:

$$G = 6.67 \times 10^{-11}\ N\cdot m^2/kg^2$$

Historical Significance:

Cavendish's experiment called "weighing the Earth" because it enabled calculation of Earth's mass using g and RE

7.5 ACCELERATION DUE TO GRAVITY OF EARTH

Modeling Earth as Concentric Shells:

For Point Outside Earth:

- Point lies outside all spherical shells
- Each shell acts as if mass concentrated at center
- Total effect: entire Earth mass acts from center
- Force: $F = GM_em/r^2$

For Point Inside Earth:

Consider point at distance r from center:

- Shells with radius > r: contribute zero force (point inside shell)
- Shells with radius ≤ r: act as if mass concentrated at center
- Only inner sphere of radius r contributes to force

Mass Relationships:

Assuming uniform density ρ:

$$M_e = (4/3)\pi R_e^3 \rho$$
$$Mr = (4/3)\pi r^3 \rho$$

Therefore:

$$Mr/M_e = r^3/R_e^3$$

Force Inside Earth:

$$F(r) = GMrm/r^2 = GM_e mr/R_e^3$$

Acceleration Due to Gravity:

At Earth's Surface:

$$g = GM_e/R_e^2$$

Inside Earth (at distance r from center):

$$g(r) = GM_e r/R_e^3 = g(r/R_e)$$

Key Point: g varies linearly with distance from center inside Earth

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7.6 ACCELERATION DUE TO GRAVITY ABOVE AND BELOW SURFACE

Above Earth's Surface:

At Height h:

$$g(h) = GM_e/(R_e + h)^2$$

For Small Heights ($h << R_e$):

Using binomial approximation:

$$g(h) \approx g(1 - 2h/R_e)$$

Interpretation: Gravity decreases by factor $(1 - 2h/R_e)$ for small heights

General Form:

$$g(h) = g/(1 + h/R_e)^2$$

Below Earth's Surface:

At Depth d:

Distance from center = $R_e - d$
Only inner sphere of radius $(R_e - d)$ contributes

Mass of Inner Sphere:

$$Ms = M_e((R_e - d)/R_e)^3$$

Gravitational Force:

$$F(d) = GMsm/(R_e - d)^2$$

Simplification:

$$g(d) = GM_e(R_e - d)/R_e^3 = g(1 - d/R_e)$$

Summary of Variations:

- **Above surface**: g decreases as $1/(1 + h/R_e)^2$
- **Below surface**: g decreases linearly as $(1 - d/R_e)$
- **Maximum g**: occurs at Earth's surface
- **At center**: $g = 0$

7.7 GRAVITATIONAL POTENTIAL ENERGY

Conservative Nature of Gravity:

- Work done by gravitational force is path-independent
- Potential energy function can be defined
- Energy conservation applies

Near Earth's Surface:

For small heights ($h << R_e$), gravitational force ≈ constant = mg

Work Done:

$$W_{12} = mg(h_2 - h_1)$$

Potential Energy:

$$U(h) = mgh + U_0$$

Where U_0 = constant (potential energy at reference level)

For Arbitrary Distances from Earth:

Gravitational Force:

$$F = GM_e m/r^2$$

Work Done (from r_1 to r_2):

$$W_{12} = \int[r_1 \text{ to } r_2]\, F\, dr = \int[r_1 \text{ to } r_2]\, (GM_e m/r^2)\, dr$$
$$W_{12} = GM_e m(1/r_1 - 1/r_2)$$

Potential Energy:

$$U(r) = -GM_e m/r + U_1$$

Where U_1 = constant

Standard Convention:

Setting $U(r \to \infty) = 0$:

$$U(r) = -GM_e m/r$$

Physical meaning: Potential energy is negative, representing bound state

General Two-Body System:

$$U = -Gm_1m_2/r$$

System of Multiple Particles:

Total potential energy = sum over all pairs:

$$U\_total = \Sigma_{i<j}(-Gm_im_j/r_{ij})$$

7.8 ESCAPE SPEED

Problem Setup:

What minimum initial speed allows object to reach infinity?

Energy Conservation Approach:

Initial Energy (at surface):

$$E_i = ½mv_i^2 - GM_e m/R_e$$

Final Energy (at infinity):

$$E∞ = ½mv∞^2 + 0$$

Conservation Condition:

$$\frac{1}{2}mv_i^2 - GM_em/R_e = \frac{1}{2}mv_\infty^2$$

Minimum Escape Speed:

For minimum v_i, set $v_\infty = 0$:

$$\frac{1}{2}mv_i^2 = GM_em/R_e$$
$$v_e = \sqrt{(2GM_e/R_e)}$$

Using Surface Gravity:

Since $g = GM_e/R_e^2$:

$$v_e = \sqrt{(2gR_e)}$$

Numerical Value:

With $g \approx 9.8$ m/s^2, $R_e \approx 6.4 \times 10^6$ m:

$$v_e \approx 11.2 \text{ km/s}$$

From Height h:

$$v_e(h) = \sqrt{(2GM_e/(R_e + h))}$$

Applications:

Moon's Escape Speed:

$$v_{moon} = \sqrt{2GM_{moon}/R_{moon}} \approx 2.3 \text{ km/s}$$

Much smaller than Earth's escape speed

Atmospheric Retention:

- Gas molecules with speeds > escape speed can leave atmosphere
- Moon has no atmosphere due to low escape speed
- Lighter molecules (H_2, He) escape more easily from Earth

7.9 EARTH SATELLITES

Satellite Motion:

Earth satellites follow same laws as planetary motion (Kepler's laws apply)

Circular Orbits:

Forces:

- **Centripetal force needed**: $F_c = mv^2/(R_e + h)$
- **Gravitational force provided**: $F_g = GM_e m/(R_e + h)^2$

Equilibrium Condition:

$$mv^2/(R_e + h) = GM_e m/(R_e + h)^2$$

Orbital Speed:

$$v = \sqrt{GM_e/(R_e + h)}$$

Key Point: Speed decreases with increasing orbital radius

Speed at Surface (h = 0):

$$v_0 = \sqrt{GM_e/R_e} = \sqrt{gR_e} \approx 7.9 \text{ km/s}$$

Orbital Period:

Time for One Revolution:

$$T = 2\pi(R_e + h)/v = 2\pi\sqrt{(R_e + h)^3/GM_e}$$

Kepler's Third Law:

$$T^2 = (4\pi^2/GM_e)(R_e + h)^3 = K(R_e + h)^3$$

Where $K = 4\pi^2/GM_e$

Low Earth Orbit:

For $h << R_e$:

$$T_0 = 2\pi\sqrt{R_e^3/GM_e} = 2\pi\sqrt{R_e/g}$$

Numerical Value:

$$T_0 \approx 85 \text{ minutes}$$

Moon as Earth Satellite:

- Orbital period ≈ 27.3 days
- Nearly circular orbit
- Rotational period ≈ orbital period (synchronous rotation)

Artificial Satellites:

Since 1957, used for:

- Telecommunications
- Weather monitoring
- Navigation (GPS)
- Scientific research
- Earth observation

7.10 ENERGY OF AN ORBITING SATELLITE

Circular Orbit Energy Analysis:

Kinetic Energy:

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m(GM_e/(R_e + h)) = GM_em/2(R_e + h)$$

Potential Energy:

$$U = -GM_em/(R_e + h)$$

Total Energy:

$$E = K + U = GM_em/2(R_e + h) - GM_em/(R_e + h)$$
$$E = -GM_em/2(R_e + h)$$

Key Relationships:

- **Total energy is negative** (bound system)
- **$|U| = 2|K|$** (virial theorem for inverse-square force)
- **$E = -K$** (total energy equals negative kinetic energy)

Energy Significance:

Negative Total Energy:

- Indicates bound orbit (satellite cannot escape)
- Energy must be supplied to increase orbital radius
- Energy required for escape = $|E|$

Energy Changes:

- **To increase orbit**: Must supply energy
- **Orbital decay**: Energy lost (due to atmospheric drag, etc.)

Elliptical Orbits:

- Kinetic and potential energies vary with position
- Total energy remains constant (conservative system)
- Total energy still negative
- $E = -GM_em/2a$ (where a = semi-major axis)

Transfer Orbits:

Energy required to change from orbit 1 to orbit 2:

$$\Delta E = E_2 - E_1 = GM_em/2(1/r_1 - 1/r_2)$$

IMPORTANT FORMULAS SUMMARY

Universal Gravitation:

- **Force**: $F = Gm_1m_2/r^2$
- **Potential Energy**: $U = -Gm_1m_2/r$
- **G**: 6.67×10^{-11} N·m²/kg²

Earth's Gravity:

- **Surface**: $g = GM_e/R_e^2$
- **Height h**: $g(h) \approx g(1 - 2h/R_e)$
- **Depth d**: $g(d) = g(1 - d/R_e)$

Orbital Motion:

- **Speed**: $v = \sqrt{(GM_e/r)}$
- **Period**: $T = 2\pi\sqrt{(r^3/GM_e)}$
- **Kepler's 3rd**: $T^2 = (4\pi^2/GM_e)r^3$

Energy:

- **Escape speed**: $v_e = \sqrt{(2GM_e/R_e)}$
- **Orbital energy**: $E = -GM_em/2r$

- **Kinetic energy**: $K = GM_em/2r$
- **Potential energy**: $U = -GM_em/r$